

Exercise 7.2.1

19-1-2020

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\begin{bmatrix} x-5 & 18 & 32 \\ 0 & x-5 & -4 \\ -2 & 5 & x+11 \end{bmatrix}$$

$$\det \begin{pmatrix} x-5 & 18 & 32 \\ 0 & x-5 & -4 \\ -2 & 5 & x+11 \end{pmatrix} = 0$$

$$(x-5) \left[(x-5)(x+11) + 20 \right] - 18(-8) + 32(+2x-10) = 0$$

$$(x-5)(x^2 + 6x - 55 + 20) + 144 + 64x - 320 = 0$$

$$x^3 + 6x^2 - 55x + 20 - 5x^2 - 30x + 275 - 100 + 144 + 64x - 320 = 0$$

$$x^3 + x^2 - x - 1 = 0$$

$$(x-1)(x+1)(x+1) = 0$$

$$x_1 = 1 \quad x_2 = -1 \quad x_3 = -1$$

$$\lambda_1 = 1 \quad \lambda_2 = \lambda_3 = -1$$

for $\lambda_1 = 1$

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} -4 & 18 & 32 \\ 0 & -4 & -4 \\ -2 & 5 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 18 & 32 \\ 0 & -4 & -4 \\ -2 & 5 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2R_3 - R_1 \rightarrow R_3$$

$$\begin{bmatrix} -4 & 18 & 32 & | & 0 \\ 0 & -4 & -4 & | & 0 \\ 0 & -8 & -8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 9 & 16 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$R_1 - 9R_2 \rightarrow R_1$$

$$\begin{bmatrix} -2 & 0 & 7 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

infinite solutions

$$z = t \quad -2x + 7z = 0$$

$$y + z = 0 \quad -2x = -7z$$

$$y = -t \quad x = \frac{7}{2}z = \frac{7}{2}t$$

$$x_1 = \begin{bmatrix} 7/2 \\ -1 \\ 1 \end{bmatrix} t$$

$$\lambda_2 = -1 \quad (-I - A)x = 0$$

$$\begin{bmatrix} -6 & 18 & 32 \\ 0 & -6 & -4 \\ -2 & 5 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 18 & 32 & | & 0 \\ 0 & -6 & -4 & | & 0 \\ -2 & 5 & 10 & | & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$3R_3 - R_1 \rightarrow R_3$$

$$\begin{bmatrix} -6 & 18 & 32 & | & 0 \\ 0 & -6 & -4 & | & 0 \\ 0 & -3 & -2 & | & 0 \end{bmatrix}$$

$$2R_3 - R_2 \rightarrow R_3$$

$$\begin{bmatrix} -6 & 18 & 32 & | & 0 \\ 0 & -6 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 10/3 \\ -2/3 \\ 1 \end{bmatrix} t$$

$$3R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} -6 & 0 & 20 \\ 0 & -6 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$z = t \quad -6x + 20z = 0$$

$$y = -\frac{2}{3}t \quad x = \frac{20}{6}t = \frac{10}{3}t$$

Exercise 7.2.2

Exercise 7.2.2.

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\det \left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} \lambda + 13 & 28 & -28 \\ -4 & \lambda - 9 & 8 \\ 4 & 8 & \lambda - 9 \end{bmatrix} = 0$$

$$4 \begin{bmatrix} 28 & -28 \\ \lambda - 9 & 8 \end{bmatrix} - 8 \begin{bmatrix} \lambda + 13 & -28 \\ -4 & 8 \end{bmatrix} + (\lambda - 9) \begin{bmatrix} \lambda + 13 & 28 \\ -4 & \lambda - 9 \end{bmatrix}$$

$$= 4(224 + 28\lambda - 252) - 8(8\lambda + 104 - 112) + (\lambda - 9)[(\lambda + 13)(\lambda - 9) + 112]$$

$$= 4(28\lambda - 28) - 8(8\lambda - 8) + (\lambda - 9)[\lambda^2 - 9\lambda + 13\lambda - 117 + 112]$$

$$= 112(\lambda - 1) - 64(\lambda - 1) + (\lambda - 9)(\lambda^2 + 4\lambda - 5)$$

$$= 112(\lambda - 1) - 64(\lambda - 1) + (\lambda - 9)(\lambda - 1)(\lambda + 5)$$

$$= (\lambda - 1)[112 - 64 + (\lambda - 9)(\lambda + 5)]$$

$$= (x-1) [48 + x^2 + 5x - 9x - 45]$$

$$= (x-1) (x^2 - 4x + 3) = (x-1)(x-3)(x-1) = 0$$

$$x=1; x=1; x=3 \quad \text{eigenvalues.}$$

$$(xI - A) = 0; \quad x=1$$

$$\left(1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 28 & -28 \\ -4 & -8 & 8 \\ 4 & 8 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} 7R_2 + 2R_1 \\ 7R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 14 & 28 & -28 & 0 \\ -4 & -8 & 8 & 0 \\ 4 & 8 & -8 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 14 & 28 & -28 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x = -2y + 2z$$

$$y = s$$

$$z = t$$

$$\begin{bmatrix} -2s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t$$

$$(\lambda I - A) = 0; \lambda = 3$$

$$\left(3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 28 & -28 \\ -4 & -6 & 8 \\ 4 & 8 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} 4R_2 + R_1 \\ 4R_3 - R_1 \end{array} \begin{bmatrix} 16 & 28 & -28 & 0 \\ -4 & -6 & 8 & 0 \\ 4 & 8 & -6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 16 & 28 & -28 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 16 & 28 & -28 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$z = t$$

$$\begin{bmatrix} \frac{7}{2}t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -1 \\ 1 \end{bmatrix} t$$

$$\text{eigen vector: } \begin{bmatrix} \frac{7}{2} \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Exercice 7.2.3

$$3) \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\begin{bmatrix} \lambda - 89 & -38 & -268 \\ -14 & \lambda - 2 & -40 \\ 30 & 12 & \lambda + 90 \end{bmatrix} = 0$$

$$\begin{aligned} & (\lambda - 89)[(\lambda - 2)(\lambda + 90) + (12)(40)] + 38[-14(\lambda + 90) + 1200] \\ & - 268[-14(12) - 30(\lambda - 2)] = 0 \end{aligned}$$

$$\begin{aligned} & (\lambda - 89)(\lambda^2 + 90\lambda - 2\lambda - 180 + 480) + 38(-14\lambda - 1260 + 1200) \\ & - 268(-168 - 30\lambda + 60) = 0 \end{aligned}$$

$$(\lambda - 89)(\lambda^2 + 88\lambda + 300) + 38(-14\lambda - 60) - 268(-30\lambda - 108) = 0$$

$$\begin{aligned} & \lambda^3 + 88\lambda^2 + 300\lambda - 89\lambda^2 - 7832\lambda - 26700 \\ & - 532\lambda - 2280 + 8040\lambda + 28944 = 0 \end{aligned}$$

$$\lambda^3 - \lambda^2 - 24\lambda - 36 = 0$$

$$\lambda_1 = 6 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

$$\lambda_1 = 6 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

$$\det(\lambda I - A) = 0$$

$$\det \left(6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -83 & -38 & -268 \\ -14 & 4 & -40 \\ 30 & 12 & 96 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ -14 & 4 & -40 & 0 \\ 30 & 12 & 96 & 0 \end{array} \right]$$

$$83R_2 - 14R_1$$

$$83R_3 + 30R_1$$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ 0 & 864 & 432 & 0 \\ 0 & -144 & -72 & 0 \end{array} \right]$$

Row echelon form

$$6R_3 + R_2$$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ 0 & 864 & 432 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$z = t$$

$$y = -\frac{1}{2}t$$

$$x = -3t$$

$$\det(-2I - A) = 0$$

$$\det \left(2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -91 & -38 & -268 \\ -14 & -4 & -40 \\ 30 & 12 & 88 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$13R_2 - 2R_1$$

$$91R_3 + 30R_1$$

$$\begin{bmatrix} -91 & -38 & -268 & | & 0 \\ -14 & -4 & -40 & | & 0 \\ 30 & 12 & 88 & | & 0 \end{bmatrix}$$

$$2R_2 + R_3$$

$$\begin{bmatrix} -91 & -38 & -268 & | & 0 \\ 0 & 24 & 16 & | & 0 \\ 0 & -48 & -32 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -91 & -38 & -268 & | & 0 \\ 0 & 12 & 8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$12y = -8t \Rightarrow y = -\frac{2}{3}t$$

$$z = t$$

$$-91x = 38\left(-\frac{2}{3}\right)t + 268t$$

$$\Rightarrow x = -\frac{8}{3}t$$

$$\begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

$$\det(-3I - A) = 0$$

$$\det \left(-3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -92 & -38 & -268 \\ -14 & -5 & -40 \\ 30 & 12 & 87 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$$+13R_2 - 2R_1$$~~
~~$$91R_3 - 30$$~~

$$\left[\begin{array}{ccc|c} -92 & -38 & -268 & 0 \\ -14 & -5 & -40 & 0 \\ 30 & 12 & 87 & 0 \end{array} \right] \begin{array}{l} 46R_2 - 7R_1 \\ 46R_3 + 15R_1 \end{array}$$

$$2R_3 + R_2$$

$$\left[\begin{array}{ccc|c} -92 & -38 & -268 & 0 \\ 0 & 36 & 36 & 0 \\ 0 & -18 & -18 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} -92 & -38 & -268 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} -5/2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{array}{l} z = t \\ y = -t \\ x = -5/2 \end{array}$$

Exercise 7.2.4

$$\det (xI - A) = 0.$$

$$\det \begin{bmatrix} x-1 & -90 & 0 \\ 0 & x+2 & 0 \\ -3 & -89 & x+2 \end{bmatrix} = 0$$

$$(x-1)(x+2)^2 + 90(0) + 0 = 0.$$

$$(x-1)(x+2)^2 = 0.$$

$$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = -2 \quad \left(\begin{array}{l} -2 \text{ is a root} \\ \text{of multiplicity } 2. \end{array} \right)$$

For $\lambda_1 = 1$: $AX = X$

$$(I - A)X = 0.$$

$$\begin{bmatrix} 0 & -90 & 0 \\ 0 & 3 & 0 \\ -3 & -89 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 0 & -90 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & -89 & 3 & 0 \end{array} \right] \quad \text{Switch row 1 and row 3.}$$

$$\left[\begin{array}{ccc|c} -3 & -89 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -90 & 0 & 0 \end{array} \right]$$

$$30R_2 + R_3 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -3 & -89 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{Divide 1st row by } (-3). \\ \text{Divide 2nd row by } 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{89}{3} & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_1 - \frac{89}{3}R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{z = t}$$

$$\boxed{y = 0}$$

$$x - z = 0 \Rightarrow \boxed{x = t}$$

solution:

$$t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_2 = \lambda_3 = -2$:

$$(-2I - A)X = 0.$$

$$\begin{bmatrix} -3 & -90 & 0 \\ 0 & 0 & 0 \\ -3 & -89 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -3 & -90 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -89 & 0 & 0 \end{array} \right]$$

Switch rows 2 and 3.

$$\left[\begin{array}{ccc|c} -3 & -90 & 0 & 0 \\ -3 & -89 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$R_2 - R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} -3 & -90 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$z = t$$

$$y = 0$$

$$-3x - 90y = 0 \Rightarrow x = 0.$$

$$X_2 = X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A is not diagonalizable since it has only 2 eigenvectors.

Exercise 7.2.5

Exercise 7.2.5:

$$\det (xI - A) = 0.$$

$$\det \begin{bmatrix} x-11 & -45 & -30 \\ -10 & x-26 & -20 \\ 20 & 60 & x+44 \end{bmatrix} = 0.$$

$$(x-11) [(x-26)(x+44) + 1200] + 45(-10x - 440 + 400) - 30(-600 - [20(x-26)]) = 0.$$

$$(x-11)(x^2 + 44x - 26x - 1144 + 1200) + 45(-10x - 40) - 30(-600 - 20x + 520) = 0.$$

$$(x-11)(x^2 + 18x + 56) - 450x - 1800 + 18000 + 600x - 15600 = 0$$

$$x^3 + 18x^2 + 56x - 11x^2 - 198x - 616 + 150x + 600 = 0.$$

$$x^3 + 7x^2 + 81x - 16 = 0.$$

$$x_1 = 1$$

$$x_2 = -4$$

$$x_3 = -4$$

(-4 is a root of multiplicity 2).

for $x_1 = 1$ $(I - A)x = 0$.

$$\begin{bmatrix} -10 & -45 & -30 \\ -10 & -25 & -20 \\ 20 & 60 & 45 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ -10 & -25 & -20 & 0 \\ 20 & 60 & 45 & 0 \end{array} \right]$$

$$R_2 - R_1 \leftrightarrow R_2$$

$$R_3 + 2R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & -30 & 15 & 0 \end{array} \right]$$

$$20R_3 + 30R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$45 R_2 + 20 R_1 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} -200 & 0 & -150 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Divide 1st row by (-200)

Divide 2nd row by 20

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{z = t}$$

$$y + \frac{1}{2}z = 0 \Rightarrow \boxed{y = -\frac{1}{2}t}$$

$$x + \frac{3}{4}z = 0 \Rightarrow \boxed{x = -\frac{3}{4}t}$$

Solution

$$t \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

For $\lambda_2 = -4$: $(-4I - A)X = 0$

$$\begin{bmatrix} -15 & -45 & -30 \\ -10 & -30 & -20 \\ 20 & 60 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -15 & -45 & -30 & 0 \\ -10 & -30 & -20 & 0 \\ 20 & 60 & 40 & 0 \end{array} \right]$$

$$15 R_2 - 10 R_1 \leftrightarrow R_2$$

$$15 R_3 + 20 R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -15 & -45 & -30 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Divide 1st row by (-15).

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{z=t}$$

$$\boxed{y=s}$$

$$x + 3y + 2z = 0 \Rightarrow \boxed{x = -3s - 2t}$$

Solution:

$$t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [x_1 \quad x_2 \quad x_3]$$

$$P = \begin{bmatrix} -\frac{3}{4} & -3 & -2 \\ -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -4 & -12 & -8 \\ -2 & -5 & -4 \\ 4 & 12 & 9 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -4 & -12 & -8 \\ -2 & -5 & -4 \\ 4 & 12 & 9 \end{bmatrix} \begin{bmatrix} 11 & 45 & 30 \\ 10 & 26 & 20 \\ -20 & -60 & -44 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & -3 & -2 \\ -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

x_1 x_2 x_3

D: diagonal matrix

Exercise 7.2.6

$$\begin{bmatrix} 95 & 25 & 24 \\ -196 & -53 & -48 \\ -164 & -42 & -43 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} x-95 & -25 & -24 \\ 196 & x+53 & 48 \\ 164 & 42 & x+43 \end{vmatrix} = 0$$

$$\begin{aligned} & (x-95) \left[(x+53)(x+43) - 2016 \right] \\ & + 25 (196(x+43) - 7872) \\ & - 24 (8232 - 164x - 8692) = 0 \end{aligned}$$

$$\begin{aligned} & (x-95)(x^2 + 96x + 2279 - 2016) \\ & + 25(196x + 8428 - 7872) \\ & - 24(-164x - 460) = 0 \end{aligned}$$

$$\begin{aligned} & (x-95)(x^2 + 96x + 263) \\ & + 25(196x + 556) + 3936x + 11040 = 0 \\ & x^3 + 96x^2 + 263x - 95x^2 - 9120x - 24985 \\ & + 4900x + 13900 + 3936x + 11040 = 0 \end{aligned}$$

$$x^3 + x^2 - 21x - 45 = 0$$

$$x_1 = 5 \quad x_2 = -3 \quad x_3 = -3$$

$$\lambda_1 = 5 \quad \lambda_2 = \lambda_3 = -3$$

$$(5I - A)x = 0$$

$$\left[\begin{array}{ccc|c} -90 & -25 & -24 & 0 \\ 196 & 58 & 48 & 0 \\ 164 & 42 & 48 & 0 \end{array} \right]$$

$$90R_2 + 196R_1 \rightarrow R_2 \quad 90R_3 + 164R_1 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} -90 & -25 & -24 & 0 \\ 0 & 320 & -384 & 0 \\ 0 & -320 & 384 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} -90 & -25 & -24 & 0 \\ 0 & 1 & -\frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$90R_1 + 25R_2 \rightarrow 90R_1$$

$$\left[\begin{array}{ccc|c} -90 & 0 & -54 & 0 \\ 0 & 1 & -6/5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$z = t$$

$$y = \frac{6}{5}t$$

$$-90x = 54t$$

$$x = -\frac{3}{5}t$$

$$x_1 = \begin{bmatrix} -3/5 \\ 6/5 \\ 1 \end{bmatrix}$$

$$A_2 = -3$$

$$\begin{bmatrix} -98 & -25 & -24 & | & 0 \\ 196 & 50 & 48 & | & 0 \\ 164 & 42 & 40 & | & 0 \end{bmatrix}$$

$$R_2 + 2R_1 \rightarrow R_2$$

$$\begin{bmatrix} -98 & -25 & -24 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 164 & 42 & 40 & | & 0 \end{bmatrix}$$

$$98R_3 + 164R_1 \rightarrow R_3$$

$$\begin{bmatrix} -98 & -25 & -24 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & \frac{2510}{16} & -\frac{816}{16} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -98 & -25 & -24 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -98 & 0 & -49 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$2 = t$$

$$y = 2 = t$$

$$\begin{bmatrix} 2 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$2x = -2$$

$$x = -\frac{t}{2}$$

$$x_2 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 7.3.1

$$\det (xI - A) = 0.$$

$$\det \left(x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) = 0.$$

$$\det \begin{bmatrix} x-1 & -2 \\ -2 & x-1 \end{bmatrix} = 0.$$

$$(x-1)(x-1) - (-2(-2)) = 0.$$

$$x^2 - 2x + 1 - 4 = 0.$$

$$x^2 - 2x - 3 = 0.$$

$$\lambda_1 = 3 \quad \lambda_2 = -1$$

for $\lambda_1 = 3$: $AX = 3X$

$$(3I - A)X = 0$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Augmented Matrix: $\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right]$

$$R_2 + R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \text{Divide 1st row by (2)}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\boxed{y = t}$$

$$x - y = 0 \Rightarrow \boxed{x = t}$$

Solution:

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -1$: $AX = -X$.

$$(-I - A)X = 0.$$

$$\left(- \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Augmented
Matrix:

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right]$$

$$R_2 - R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\boxed{y = s}$$

$$-2x - 2y = 0$$

$$-2x = 2s \Rightarrow \boxed{x = -s}$$

Solution:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} s \Rightarrow X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Augmented Matrix: $\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$

$$R_2 - R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array} \right]$$

$$2R_2 + R_1 \leftrightarrow R_1$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \end{array} \right] \quad \begin{array}{l} \text{Divide 1st row and} \\ \text{2nd row by 2} \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array} \right]$$

$\underbrace{\hspace{10em}}_{P^{-1}}$

Then,

$$P^{-1}AP$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \lambda_1 \leftarrow \\ = \end{array} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{Diagonal matrix: } D} \begin{array}{c} \rightarrow \lambda_2 \end{array}$$

Diagonal matrix: D .

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore,

$$A^{10} = P D^{10} P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 3^{10} & -1 \\ 3^{10} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow A^{10} = \begin{bmatrix} \frac{3^{10}+1}{2} & \frac{3^{10}-1}{2} \\ \frac{3^{10}-1}{2} & \frac{3^{10}+1}{2} \end{bmatrix}$$

Exercise 7.3.2

Lx: 7.3.2 $A = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix}$

eigen values: $\det(xI - A) = 0$

$$\begin{vmatrix} x-1 & -4 & -1 \\ 0 & x-2 & -5 \\ 0 & 0 & x-5 \end{vmatrix} = 0 \Rightarrow (x-1)(x-2)(x-5) = 0$$

$\lambda_1 = 5$; $\lambda_2 = 2$; $\lambda_3 = 1$

for $\lambda_1 = 5$ $(\lambda_1 I - A)(x) = 0$

$$\left(\begin{array}{ccc|c} 4 & -4 & -1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$\underline{z = t} \Rightarrow y = \underline{\frac{5}{3}t}$; $x - \frac{5}{3}t - \frac{1}{4}t = 0 \Rightarrow x = \underline{\underline{\frac{23}{12}t}}$

$\underline{\underline{X_1 = \begin{pmatrix} \frac{23}{12} \\ \frac{5}{3} \\ 1 \end{pmatrix}}}$

for $\lambda_2 = 2$ $\left(\begin{array}{ccc|c} 1 & -4 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$

$\underline{\underline{z = 0}}$ $\underline{\underline{y = t}}$ $x - 4t = 0 \Rightarrow x = \underline{\underline{4t}}$

$\underline{\underline{X_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}}}$

* for $\lambda_3 = 1$

$$\left(\begin{array}{ccc|c} 0 & -4 & -1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right) R_2 \rightarrow 4R_2 - R_1 \rightarrow$$

$$\left(\begin{array}{ccc|c} 0 & -4 & -1 & 0 \\ 0 & 0 & -19 & 0 \\ 0 & 0 & -4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

S_0

$$P = \begin{pmatrix} \frac{93}{12} & 4 & 1 \\ \frac{5}{3} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{5}{3} \\ 1 & -4 & \frac{19}{4} \end{pmatrix}$$

$$*D = P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{5}{3} \\ 1 & -4 & \frac{19}{4} \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{23}{12} & 4 & 1 \\ \frac{5}{3} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$*A^{50} = PD^{50}P^{-1} = \begin{pmatrix} \frac{93}{12} & 4 & 1 \\ \frac{5}{3} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{5}{3} \\ 1 & -4 & \frac{19}{4} \end{pmatrix}$$

Exercise 7.3.3

Ex 7.3.3 $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$

eigen values:

$$\det(xI - A) = 0 \Rightarrow \begin{vmatrix} x-1 & 2 & -1 \\ -2 & x+1 & -1 \\ 2 & -3 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow (x-1) [(x+1)(x-1) - 3] - 2[(-2)(x-1) + 2] + [6 - 2(x+1)] = 0$$

$$(x-1)(x^2 - 1 - 3) - 2(-2x + 2 + 2) + (6 - 2x - 2) = 0$$

$$\Rightarrow (x-1)(x^2 - 4) - 2(-2x + 4) + (-2x + 4) = 0$$

$$\Rightarrow x^3 - 4x - x^2 + 4 + 4x - 8 - 2x + 4 = 0$$

$$\Rightarrow x^3 - x^2 - 2x = 0 \quad \lambda_1 = \underline{\underline{2}} \quad \lambda_2 = \underline{\underline{-1}} \quad \lambda_3 = \underline{\underline{0}}$$

* for $\lambda_1 = 2$, $(2I - A)(x) = 0$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ -2 & 3 & -1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \sim \begin{pmatrix} 1 & 2 & -1 & | & 0 \\ -2 & 3 & -1 & | & 0 \\ 2 & -3 & 1 & | & 0 \end{pmatrix} \begin{matrix} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & 7 & 1 & | & 0 \\ 0 & -7 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & 7 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & 1 & \frac{1}{7} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\cdot z = t \Rightarrow y = -\frac{1}{7}t \Rightarrow x - \frac{8}{7}t + t = 0 \Rightarrow x = -\frac{5}{7}t$$

$$x_1 = \begin{pmatrix} -\frac{5}{7} \\ -\frac{1}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 7 \end{pmatrix}$$

• for $\lambda_2 = -1$ $(-I - A)X = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ -2 & 0 & -1 & 0 \\ 2 & -3 & -2 & 0 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_1 - R_2 \\ R_3 \rightarrow R_1 + R_3 \end{array} \rightarrow$$

$$\left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right) \begin{array}{l} R_3 \rightarrow R_2 - 2R_3 \end{array} \rightarrow \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \underline{\text{So}} \quad \underline{z = t}, \underline{y = -t} \text{ \& } \underline{x = -\frac{1}{2}t}$$

$$X_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} x^2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

• for $\lambda_3 = 0$ $(0I - A)(X) = 0$

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -3 & -1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \underline{\text{So}} \quad \underline{z = t} \quad \underline{y = -t} \text{ \& } \underline{x = -t}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -5 & -1 & -1 \\ 1 & -2 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$* P^{-1} = ?$$

$$\det(P) = -6 \quad \text{Cof}(P) = \begin{pmatrix} 0 & -6 & 12 \\ -1 & 2 & 3 \\ 1 & -4 & 9 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{3} & \frac{3}{6} \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{3} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -5 & -1 & -1 \\ -1 & -2 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow A^{100} = P D^{100} P^{-1} = \begin{pmatrix} -5 & -1 & -1 \\ 1 & -2 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 & 0 \\ 0 & -1^{100} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{3} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -1 & -\frac{5}{6} \times 2^{100} + \frac{1}{3} & -\frac{5}{6} \times 2^{100} - \frac{2}{3} \\ -2 & -\frac{1}{6} \times 2^{100} + \frac{2}{3} & -\frac{1}{6} \times 2^{100} - \frac{1}{3} \\ 2 & \frac{2}{6} \times 2^{100} - \frac{2}{3} & \frac{2}{6} \times 2^{100} + \frac{1}{3} \end{pmatrix}$$

Exercise 7.4.1

Exercise 7.4.1:

$$\det (xI - A) = 0.$$

$$\det \begin{bmatrix} x-11 & 1 & 4 \\ 1 & x-11 & 4 \\ 4 & 4 & x-14 \end{bmatrix} = 0.$$

$$(x-11) [(x-11)(x-14) - 16] - [x-14 - 16] + 4 [4 - (4(x-11))] = 0.$$

$$(x-11) (x^2 - 14x - 11x + 154 - 16) - x + 30 + 4(4 - 4x + 44) = 0.$$

$$(x-11) (x^2 - 25x + 138) - x + 30 + 4(-4x + 48) = 0.$$

$$x^3 - 25x^2 + 138x - 11x^2 + 275x - 1518 - x + 30 - 16x + 192 = 0.$$

$$x^3 - 36x^2 + 396x - 1296 = 0.$$

$$\lambda_1 = 6, \quad \lambda_2 = 19, \quad \lambda_3 = 12$$

For $\lambda_1 = 6$: $(6I - A)X = 0.$

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -5 & 4 \\ 4 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 1 & -5 & 4 & 0 \\ 4 & 4 & -8 & 0 \end{array} \right]$$

$$5R_2 + R_1 \leftrightarrow R_2$$

$$5R_3 + 4R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 24 & -24 & 0 \end{array} \right]$$

$$R_3 + R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$24 R_1 + R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} -120 & 0 & 120 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \text{Divide 1st row by } (-120) \\ \text{Divide 2nd row by } (-24) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{z = t}$$

$$y - z = 0 \Rightarrow \boxed{y = t}$$

$$x - z = 0 \Rightarrow \boxed{x = t}$$

$$\text{eigenvector: } x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\text{for } \lambda_2 = 18: (18I - A)X = 0$$

$$\begin{bmatrix} 7 & 1 & 4 \\ 1 & 7 & 4 \\ 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 1 & 7 & 4 & 0 \\ 4 & 4 & 4 & 0 \end{array} \right]$$

$$7R_2 - R_1 \leftrightarrow R_2$$

$$7R_3 - 4R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 0 & 49 & 24 & 0 \\ 0 & 24 & 12 & 0 \end{array} \right]$$

$$2R_3 - R_2 \leftrightarrow R_3.$$

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 0 & 49 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$49 R_1 - R_2 \leftrightarrow R_1.$$

$$\left[\begin{array}{ccc|c} 336 & 0 & 168 & 0 \\ 0 & 49 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \text{Divide 1st row by 336} \\ \text{Divide 2nd row by 49} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{z = t}$$

$$y + \frac{1}{2}z = 0 \Rightarrow \boxed{y = -\frac{1}{2}t}$$

$$x + \frac{1}{2}z = 0 \Rightarrow \boxed{x = -\frac{1}{2}t}$$

$$X_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

for $\lambda_3 = 12$: $(12I - A)X = 0$.

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 1 & 1 & 4 & 0 \\ 4 & 4 & -2 & 0 \end{array} \right]$$

$$R_2 - R_1 \leftrightarrow R_1$$

$$R_3 - 4R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & 0 \end{array} \right]$$

Divide 3rd row by (-18).

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\boxed{y = t}$$

$$\boxed{z = 0}$$

$$x + y + 4z = 0 \Rightarrow \boxed{x = -t}$$

$$x_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \\ -3\sqrt{6} & -3\sqrt{6} & 6\sqrt{6} \\ -6\sqrt{2} & 6\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$\begin{matrix} \lambda_1 \leftarrow & \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 12 \end{bmatrix} & \begin{matrix} \rightarrow \lambda_2 \\ \rightarrow \lambda_3 \end{matrix} \end{matrix}$$

which is the desired diagonal matrix

Exercise 7.4.2

7.4.2.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix} \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 4 & -1 & 2 \\ -1 & \lambda - 4 & 2 \\ 2 & 2 & \lambda - 7 \end{vmatrix} = 0$$

$$(\lambda - 4)[(\lambda - 4)(\lambda - 7) - 4] + [(-\lambda + 7) - 4] + 2(-2 - 2\lambda + 8) = 0$$

$$(\lambda - 4)(\lambda^2 - 11\lambda + 28 - 4) - \lambda + 3 + 12 - 4\lambda = 0$$

$$\lambda^3 - 11\lambda^2 + 24\lambda - 4\lambda^2 + 44\lambda - 96 - 5\lambda + 15 = 0$$

$$\lambda^3 - 15\lambda^2 + 63\lambda - 81 = 0$$

$$\lambda_1 = 9 \quad \lambda_2 = 3 \quad \lambda_3 = 3$$

$$\underline{\lambda_1 = 9} \quad \underline{\lambda_2 = 3} \quad \underline{\lambda_3 = 3}$$

$$\lambda_1 = 9$$

$$(9I - A)x = 0$$

$$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 5 & 2 & 0 \\ 5 & -1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & -6 & -3 & 0 \end{array} \right]$$

$$\begin{aligned} \cdot R_2 + R_1 &\rightarrow R_2 \\ \cdot R_3 - 5R_1 &\rightarrow R_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$z = t$$

$$6y + 3z = 0$$

$$y = -\frac{1}{2}t$$

$$x - \frac{1}{2}t + t = 0$$

$$x = -\frac{1}{2}t$$

$$x_1 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

divide by $\sqrt{3/2}$

$$x_1 = \begin{bmatrix} -1/2 / \sqrt{3/2} \\ -1/2 / \sqrt{3/2} \\ 1 / \sqrt{3/2} \end{bmatrix}$$

$$\rightarrow x_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$\lambda_2 = 3$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_3 - R_2 \rightarrow R_3$$

$$2R_2 + R_1 \rightarrow R_2$$

$$z = t$$

$$y = s$$

$$2x + 2s - 4t = 0$$

$$x = 2t - s$$

$$\begin{bmatrix} 2t - s \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{divide } \sqrt{5}} t + \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{divide } \sqrt{2}} s$$

$$\lambda_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -\sqrt{2}/2\sqrt{3} & 2/\sqrt{5} & -1/\sqrt{2} \\ -\sqrt{2}/2\sqrt{3} & 0 & 1/\sqrt{2} \\ \sqrt{2}/3 & 1/\sqrt{5} & 0 \end{bmatrix} \quad P^T = \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$